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Assessment of the Profitability of a Revenue Investment with Random Times between Uniform Successive Returns

Nazim Noueihed[1] & Fadi Asrawi[2*]

[1] Department of Mathematics, Division of Mathematical Sciences, Faculty of Arts & Sciences, Haigazian University, Lebanon

[2] Faculty of Business Administration and Economics, Haigazian University, Lebanon

*Correspondence: Fadi Asrawi, Professor, Faculty of Business Administration and Economics, Haigazian University, Mexique Street, Kantari, Beirut, Lebanon. Tel: 961-11349230; E-mail: fadi.asrawi@haigazian.edu.lb

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Abstract

Investments with random cash flow streams are much closer to describe the real financial environment of a revenue generating investment than constant cash flow streams .A cash flow stream is considered under risk if at least one of its parameters is a random variable with a given probability distribution. Many models of random cash flows were investigated by researchers. However, this study aims to survey the economic worth of cash flows with random time between equal returns on the initial investment. This study develops criteria for the profitability based on the rate of return of the expected net present worth, and make comparisons to the expected rate of return based on simulations. The concept of moment generating functions of random variables is employed to get the analytic forms of the expectations. This study also considers many possible distributions for the random separating time between returns and supply a sufficient number of numerical examples. The findings show that this financial model applies to apartment buildings, condominiums, shopping malls, and the management of inventory stocks of expensive items.

JEL Classifications: C0, C13

Keywords: investment, cash flow, discount rate, present value, random variables, expectations, moment generating functions

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Definitions of Technical Symbols

P: Initial size of the investment

A: the value of uniform return

n: number of receipts of equal value “A”

d: hurdle rate, or minimum attractive rate of return

PW: net present worth

AW: net annual worth

FW: net future worth

B/C: benefit – cost ratio

ROR: rate of return

M (t): Moment generating function of a random variable

X: separation time between two successive receipts

U (a, b): uniform probability distribution on the interval (a, b)

N (μ, σ): normal distribution with mean μ and standard deviation σ

Exp. (a): exponential distribution with parameter “a”

1. Introduction

When one of the parameters in a cash flow stream is considered to be random such as annual costs, initial cost, salvage value, annual returns or the useful life, a revenue generating investment is considered to be under risk. Several economic analysis methods exist to deal with random investments and measure profitability under risk. Few examples include the net present worth, annual worth, benefit-cost ratio and the rate of return. The expectations of the net present worth and annual worth could be calculated by assuming certain independence conditions and decisions could be made where the investment with positive expectation is accepted. The Monte-Carlo simulation is very useful to find the expected rate of return because as the root of an equation with random coefficients, the distribution of the rate of return is very difficult to find. The rate of return of the expected net present worth equation could be used as another indicator. The distribution of the benefit cost-ratio is tough to find, and in some cases there is no expected value.

The applicability of the economic analysis of investments under risk depends on the degree of reality correspondence with the investment model. In this paper we will be considering an investment where the only random element is the separation time between two successive returns, and the returns are equal in value. This suggested model has many real life applications: Apartment buildings or condominiums, shopping malls, and the management of inventory models with slowly moving items or expensive items where the time value of money plays a dominant role.

Our model consists of an initial investment with a given cost “P” and a certain number “n” of equal returns of value “A” that occur at random times, that is the separation time between two successive flows is a random variable.

Our analysis leads to a compact closed form for the expected net present worth where the rest of the profitability measures can be deduced. However, for the rate of return case we shall perform simulations and compare the results to the ones obtained from the expected net present worth equation.

In order to reach a closed form for the expected net present worth, we shall adopt continuous compounding and that will allow us to use the concept of moment generating function of the separation time between returns. The criteria for the acceptability of the investment as profitable will be set in terms of present worth, annual worth, and benefit-cost ratio. Simulations will be employed to handle the case of the rate of return; we shall also consider some special distributions for the separation time and provide a sufficient number of numerical examples.

2. Literature Review

Bussey (1978) described investment projects as simple or non-simple, pure or mixed projects. Simple projects are defined as having only one sign change in the cash flow profile, the investment takes place only at the beginning of the project, and the revenues are the following cash flows. Hillier (1963) dealt with the derivation of the probability distributions of the NPV, annual cost, and the IRR in order to evaluate the risk of an investment. He reasoned that the simplified procedures that include probability reduce the estimates of each cash flow to a single expected value and fail to account for variance. Variance should also be included because it is also a decision factor. When the expected rates of return are the same, the small variance is always chosen. Some of the theoretical procedures include choosing the investments where the expected rate of return is greater than the cost of capital, sensitivity analysis and determination of the expected value of the utility. Hillier's (1963) procedure falls in between the simplified and theoretical procedures. Keeley and Westerfield (1967) found that Hiller's method results in large errors in computation because the probability of NPV misstates the value of the distant cash flows relative to the early ones. Kaplan and Barish (1967) used Hillier's approach only to calculate the PDF of IRR when the cash flows are normal random variables. They used another approach treating the minimum acceptable internal rate of return as a random variable. Bhattacharya (1978) found that the mean reverting cash flows are likely to be more relevant than the extrapolative random walk cash flows processes for sound economic reasons. In Kahraman's (2008) text two of the articles are specifically on the IRR topic using fuzzy data. Similarly, Pohjola and Turunen (1990) estimated the IRR distribution from fuzzy data. Monte Carlo simulation has been used by many authors such as Ranasinghe and Russell (1992) to find the PDF of the IRR. Other authors, such as Hertz (1964), Elnicki (1970), and Lewellen and Long (1972), used simulation to address the IRR distribution problems. Hertz (1964) used a Monte Carlo simulation method to find the PDF of IRR when the cash flows are functions of several variables each with their own probability distributions. Chen and Moore (2002) used the predicted distributions from sample and previous information about the parameters to recommend a simulation method.

In our model we will use the Monte Carlo simulation to find the expected rate of return due to the fact that the root of an equation with random coefficients, the distribution of the rate of return is difficult to find.

3. The Mathematical Model

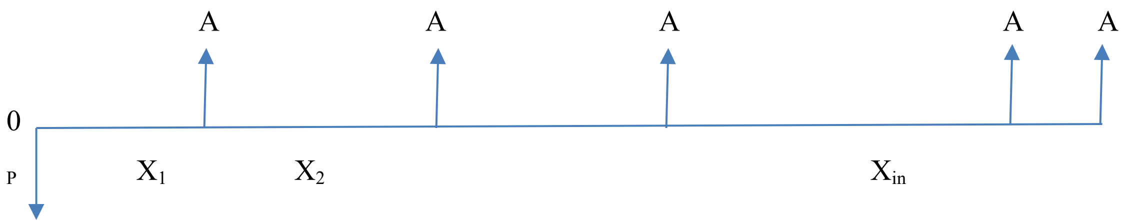
Parameter of the Investment:

P: Initial cost or size of the investment

A: Uniform return value

n: number of receipts each of value "A"

The Investment is represented by the cash flow streams:



Where "X_J" is the random variable representing the separation time between the "Jth" and the (J+1) th returns. X_1 is the separation time between 0 and the first return. Let "i" be the variable discount rate which is assumed to be a continuous real variable $-1<i<\infty$

The present worth function based on continuous compounding is:

$$PW\ (i)= -P + A\ e^{-iX1}+A\ e^{-i(X1+X2)}+\ldots+A\ e^{-i(X1+X2)}= -P+A\sum_{J=1}^{n} e^{-iY_J}$$

Where $Y_J = X_1+X_2\ldots+X_J$

and $Y_1 = X_1$

Now the variables X_1, X_2..., X_n are independent and identically distributed with a common distribution given by a single variable X.

Let M(t) be the moment generating function of X. According to the theory of generating function the moment generating function of Y_J is:

$$M_{YJ}\ (t) = M^J\ (t)$$

By the definition of generating function

$$M_{YJ}\ (t) = E\ (e^{tY_J})$$

It follows that

$$E\left(e^{-iY_J}\right)=M_Y(-i)=M^J(-i)$$

Now

$$E\left(PW(i)\right)= -P+A\sum_{J=1}^{n} E\ \left(e^{-iY_J}\right)= -P+A\sum_{J=1}^{n} M^J(-i)$$

The series represent a geometric sequence which becomes:

$$E\left(PW(i)\right)= -P=A\left[\frac{M^{(n+1)}(-i)-1}{M\ (-i)-1}-1\right]$$

Now the expected annual worth E (AW) has the same sign as E (PW) and the same in time for expected future worth E (FW). Also, since the cost is fixed at "P" the Expected benefit-cost ratio is simple:

$$\frac{E(PW)}{P}$$

So, it suffices to analyze the expected present worth expression.

The criterion we set for the profitability of the investment is to find a range of admissible value of the discount rate for which the expected present worth is positive. That is:

$$A\left[\frac{M^{(n+1)}(-i)-1}{M(-i)-1}-1\right]>P \tag{1}$$

and the average expected profitability in terms of E(PW) is:

$$\frac{A\left[\frac{M^{(n+1)}(-i)-1}{M(-i)-1}-1\right]-P}{nE(X)}$$

Where E(X) is the expected value of the reparation time.

By setting:

$$E\big(PW(i)\big)=0 \tag{2}$$

We can calculate a range of admissible values for the hurdle rate or discount rate for which the investment is profitable or attractive. It is worth mentioning that for n≥3 numerical techniques should be employed to solve the equation.

It is also important to mention that equation (2) has a unique root because n*A>P, (positive total flow).

3.1. Special Case 1

X has a uniform distribution

Example 1:

P = $5,000, A=$1,000, n=7

And X = U (1, 3)

$$M(-i)=\frac{e^{-i}-e^{-3i}}{2i}$$

Inequality (1) becomes: $1000\left(\frac{\left[\frac{e^{-i}-e^{-3i}}{2i}\right]^{3}-1}{\frac{e^{-i}-e^{-3i}}{2i}-1}-1\right)>5000$

Letting $\frac{e^{-i}-e^{-3i}}{2i}=X$

$$\Rightarrow\left(\frac{X^3-1}{X-1}-1\right)>5$$

$$\Rightarrow X>0.915$$

$$\Rightarrow\frac{e^{-i}-e^{-3i}}{2i}>0.915$$

$\Rightarrow$i<9% per period of time

Simulation gives an expected rate of return of 3.85%

Example 2:

P= \$10,000, A= \$1,500, n=10

X=U (0.5, 1.5)

$$M(-i)=\frac{e^{-0.5i}-e^{-1.5i}}{i}$$

and inequality (1) becomes: $1500\left(\frac{[\frac{e^{-0.5i}-e^{-1.5i}}{i}]^{11}-1}{\frac{e^{-0.5i}-e^{-1.5i}}{i}-1}-1\right)>10000$

Letting $\frac{e^{-0.5i}-e^{-1.5i}}{i}=X$

$$\Rightarrow\left(\frac{X^{11}-1}{X-1}-1\right)>6.66$$

$$\Rightarrow X>0.92$$

$$\Rightarrow\frac{e^{-0.5i}-e^{-1.5i}}{i}>0.92$$

$$\Rightarrow i<40\%$$

Which gives an admissible range < 90%. This is a well-accepted range for practically no MAKR value/period that exceeds 40%

Simulation gives an expected rate of return of 38%

Example 3:

P=\$3,000,000, A=\$300,000, n=10

X=U (0, 2)

$$M(-i)=\frac{1-e^{-2i}}{2i}$$

Inequality (1) becomes: $300000\left(\frac{[\frac{1-e^{-2i}}{2i}]^{10}-1}{\frac{1-e^{-2i}}{2i}-1}-1\right)>3000000$

Letting $\frac{1-e^{-2i}}{2i}=X$

$$\Rightarrow\left(\frac{X^{10}-1}{X-1}-1\right)>10$$

$$\Rightarrow X>0.94$$

$$\Rightarrow\frac{1-e^{-2i}}{2i}>0.94$$

$\Rightarrow$i<5% per period of time

Simulation gives an expected rate of return of 5.23%

As i<5%, simulation gives an acceptable range i<5.2% with 99% probability.

3.2. Special Case 2

X has an exponential distribution

Example 1:

P=$5,000, A=$1,000, n=7

X=Exp. (1)

$$M(-i)=\frac{1}{1+i}$$

and condition (1) becomes:

$$\left[\frac{\left(\frac{1}{1+i}\right)^{8}-1}{\frac{1}{1+i}-1}-1\right]>5$$

Setting $\frac{1}{1+i}=X$, the inequality becomes

$$\Rightarrow\left(\frac{X^{8}-1}{X-1}-1\right)>5$$

$$\Rightarrow X>0.915$$

Setting $\frac{1}{1+i}=0.915$

$$\Rightarrow 1+i=\frac{1}{0.915}=1.092$$

$$\Rightarrow i\approx 9.2\%$$

Which gives the admissible range for the discount rate as i<9.2%

Simulation gives an admissible range i<10% with 99% probability.

Example2:

P=$ 1,000, A=$1,500, n=10

X=Exp. (2)

$$M(-i)=\frac{2}{2+i}$$

and condition (1) becomes:

$$\left[\frac{\left(\frac{2}{2+i}\right)^{11}-1}{\frac{2}{2+i}-1}-1\right]>10$$

Letting $\frac{2}{2+i}$=X , the inequality becomes

$$\Rightarrow\left(\frac{X^4-1}{X-1}-1\right)>6.66$$

$$\Rightarrow X>0.92$$

Setting $\frac{2}{2+i}$=0.92

$$\Rightarrow i=17.3\%$$

which gives an admissible range for the discount rate as i<17.3%

Simulation gives a range of 18% with 99% probability.

Example3:

P= $3,000,000, A=$300,000 n=15

X=Exp. (0.5)

$$M(-i)=\frac{0.5}{0.5+i}$$

and condition (1) becomes:

$$\left[\frac{\left(\frac{0.5}{0.5+i}\right)^{16}-1}{\frac{0.5}{0.5+i}-1}-1\right]>10$$

Letting $\frac{0.5}{0.5+i}$=X , the inequality becomes

$$\Rightarrow\left(\frac{X^{16}-1}{X-1}-1\right)>0$$

$$\Rightarrow X>0.94$$

Now Setting $\frac{0.5}{0.5+i}$=0.94

$$\Rightarrow i=3.2\%$$

which gives the admissible range for the discount rate i<3.5%

Simulation gives an admissible range of 3.77% with 99% probability.

3.3. Special Case 3

P = $5,000, A =$1,000, n= 7

X has a normal distribution

Example 1:

X = N (1, 0.2)

In this case $M(-i)=e^{-i+0.02i^2}$

Inequality (1) becomes:

$$\left[\frac{\left(e^{-i+0.02i^2}\right)^8-1}{e^{-i+0.02i^2}-1}-1\right]>5$$

Letting $e^{-i+0.02i^2}=X$

We get

$$\left(\frac{X^8-1}{X-1}-1\right)>5$$

$\Rightarrow X>0.915$

Setting $e^{-i+0.02i^2}=0.915$

and taking the logarithm of both sides gives
$-i+0.02i^2=-0.0888$

$\Rightarrow 0.02i^2-i+0.0888=0$

which is a quadratic equation in "i"

$\Delta=1-4(0.02)(0.0888)=0.9929$

$\Rightarrow$the acceptable root: $i=\frac{1-\sqrt{0.9929}}{0.04}=0.089$

$\Rightarrow i=8.9\%$

The other roots give a value with no Economic significance.

The obtained range of admissible values for the discount rates is $i<8.9\%$

Simulation gives a range $i<9.2\%$ with 99% probability.

Example 2:

P=\$20,000, A=\$3,000, n=10

$X = N(2, 1)$

In this case $M(-i)=e^{-2i+i^2/2}$

Inequality (1) becomes:

$$\left[\frac{\left(e^{-2i+i^2/2}\right)^{11}-1}{e^{-2i+i^2/2}-1}-1\right]>6.66$$

Letting $e^{-2i+i^2/2}=X$

We get

$$\left(\frac{X^{11}\text{-}1}{X\text{-}1}\text{-}1\right)>6.66$$

$$\Rightarrow X>0.92$$

Setting $e^{-2i+i^2/2}=0.92$

and taking the logarithm of both sides gives

$$-2i+\frac{i^2}{2}=-0.0833$$

$$\Rightarrow\frac{i^2}{2}-2i+0.0833=0$$

$$\Delta=4-4\left(\frac{1}{2}\right)(0.0833)=3.8334$$

$$\Rightarrow\text{the acceptable root: } i=\frac{2-\sqrt{3.8334}}{1}=0.042$$

$$\Rightarrow i=4.2\%$$

The other root has no Economic significance.

The admissible range for the discount rates is $i<4.2\%$

Simulation gives a range of $i<4.65\%$ with 99% probability.

Example 3:

P=$1,000,000, A=$ 100,000, n=15

$X = N(3, 1)$

$$M(-i)=e^{-2i+i^2/2}$$

Inequality (1) becomes:

$$\left[\frac{\left(e^{-3i+i^2/2}\right)^{16}-1}{e^{-3i+i^2/2}-1}-1\right]>10$$

Setting $e^{-3i+i^2/2}=X$ gives:

$$\left(\frac{X^{16}\text{-}1}{X\text{-}1}\text{-}1\right)>10$$

$$\Rightarrow X>0.94$$

Setting $e^{-3i+i^2/2}=0.94$

and taking the logarithm of both sides gives

$$-3i+\frac{i^2}{2}=-0.0618$$

$$\Rightarrow\frac{i^2}{2}-3i-0.0618=0$$

$$\Delta=9-9\left(\frac{1}{2}\right)(0.0618)=8.8764$$

which give the acceptable root: $i=\frac{3-\sqrt{8.8764}}{1}=0.020$

$$\Rightarrow i=2\%$$

and the admissible range for the discount rates is $i<2\%$

Simulation gives a range $i<2.22\%$ with 99% probability.

4. Conclusion

These cases derived from this article provide solutions by which numerical and approximate methods can be compared and benchmarked. Besides, this paper has considered an investment where the only random element is the separation time between two successive returns, and the returns are equal in value.

Our analysis leads to a compact closed form for the expected net present worth where the rest of the profitability measures can be deduced. By using Monte Carlo simulations and adopting a continuous compounding, we were able to find the optimum rate of return and compare the results to the ones obtained from the expected net present worth equation. This allowed us to reach a closed form for the expected net present worth and to use the concept of moment generating function of the separation time between returns respectively.

5. Limitations and Future Suggestions

Our research focused on one essential aspect of a random investment; the separation time between successive returns. It ignored the fluctuations of the returns which may well be modelled stochastically. The paper also ignored the effect of inflation. Including the effect of inflation could lead to estimate the real value of profitability.

Future research could focus on one or both of these aspects as well as the randomness of the separation times with new real life applications.

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